DE CURVIS
IN QUIBUS SECTIONES DUARUM CURVARUM SECUNDI GRADUS, SI LEGE QUADAM MOVENTUR, SESE EXCIPIUNT.

DISSERTATIO

QUAM

AMPLISSIMI ORDINIS PHILOSOPHORUM AUCTORITATE

PRO

SUMMIS IN PHILOSOPHIA HONORIBUS RITE OBTINENDIS

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AUCTOR

MAURITIUS SADEBECK

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CUM TABULA LITHOGRAPHICA.

VRATISLAVIAE

TYPIS EX OFFICINA KUPFERIANA.

Duae curvae planae, si lege quadam in plano ita moventur, ut se intersecent, nova sectionibus sese excipientibus curva existet. Aequationes curvarum sint

$$\left.\begin{array}{l} F\ (u\ ;\ v) = o \\ F^{\prime}\ (u_{\prime}\ ;\ v_{\prime}) = o \end{array}\right\} \qquad (A.)$$

in quibus u et v, $u_{\prime}$ et $v_{\prime}$ curvarum coordinatas orthogonales esse volumus. Jam si ponimus, utrosque coordinatarum axes ab eodem puncto initium capere, atque utramque curvam in suo abscissarum axi cum celeritate constanti moveri, ordinatae quidem v et $v_{\prime}$ non mutantur, abscissae autem u et $u_{\prime}$ transeunt in $(u - mt)$ et $(u_{\prime} - nt)$, si m et n celeritatem, et t tempus significant.

Quo facto aequationes (A.) in has abeunt,

$$\left.\begin{array}{l} F\ [(u - mt);\ v] = o \\ F^{\prime}\ [(u_{\prime} - nt);\ v_{\prime}] = o \end{array}\right\} \qquad (B.)$$

et eliminata quantitate t aequatio singulas complectens sectiones prodit haec

$$\varphi\ [u;\ v;\ u_{\prime};\ v_{\prime}] = o \qquad (C.)$$

Verum ut curva nata ad unum eundemque abscissarum axem referatur, novum axem $(y = o)$ ita ponere aptum videtur, ut angulus, quem axes $(v = o)$ et $(v_{\prime} = o)$ in positiva parte includunt, et quem 2ω nominare volumus, medius intersecetur. Tunc ponendum est

$$\left\{\begin{array}{l} u = x\ \mathrm{Cos}\ \omega + y\ \mathrm{Sin}\ \omega \\ v = y\ \mathrm{Cos}\ \omega - x\ \mathrm{Sin}\ \omega \end{array}\right.$$

$$\left\{\begin{array}{l} u_{\prime} = x\ \mathrm{Cos}\ \omega - y\ \mathrm{Sin}\ \omega \\ v_{\prime} = y\ \mathrm{Cos}\ \omega + x\ \mathrm{Sin}\ \omega \end{array}\right.$$

quibus valoribus substitutis aequatio (C.) induit formam

$$\varphi\ [x\ ;\ y] = o. \qquad (D.)$$

Haec in universum disquisita si ad curvas gradus secundi conferimus, aequationes (A.) mutandae sunt in

$$\left\{\begin{array}{l} A v^2 + B u v + C u^2 + D v + E u + F = o \\ A_{\prime} v_{\prime}^2 + B_{\prime} u_{\prime} v_{\prime} + C_{\prime} u_{\prime}^2 + D_{\prime} v_{\prime} + E_{\prime} u_{\prime} + F_{\prime} = o \end{array}\right. \qquad (E)$$

et aequationes (B.) formam sumunt

$$(F.)\begin{cases} A v^2 + B v (u - mt) + C (u - mt)^2 + D v + E (u - mt) + F = o \\ A_1 v_1^2 + B_1 v_1 (u_1 - nt) + C_1 (u_1 - nt)^2 + D_1 v_1 + E_1 (u_1 - nt) + F_1 = o \end{cases}$$

unde nascitur

$$(G.)\; o = \begin{cases} n C_1 [2 C u + B v + E \mp \sqrt{v^2 (B^2 - 4AC) + 2v (BE - 2CD) + E^2 - 4CF}\,] \\ - m C [2 C_1 u_1 + B_1 v_1 + E_1 \mp \sqrt{v_1^2 (B_1^2 - 4 A_1 C_1) + 2 v_1 (B_1 E_1 - 2 C_1 D_1) + E_1^2 - 4 C_1 F_1}\,] \end{cases}$$

Tum vero coordinatis dicto modo commutatis, radicibusque remotis, inter novas coordinatas x et y aequatio quarti gradus evadit, quae tamen multis in casibus, qui postea cognoscentur, in gradum quarto inferiorem reducitur.

Jam vero ad casus singulares transire liceat; atque, ne modum excedam, eas tantum curvas, quae curvis unius generis gignantur, illustrare in animo est.

Aequationes

$$(1.)\quad \begin{cases} v^2 - au = o \\ v_1^2 - b u_1 = o \end{cases}$$

si ($v = o$; $u = o$) idem punctum atque ($v_1 = o$; $u_1 = o$) denotat, duobus parabolis in verticibus sese secantibus satisfaciunt; at, si parabolae dicta lege moventur, statim in has transeunt

$$(2.)\quad \begin{cases} v^2 - a (u - mt) = o \\ v_1^2 - b (u_1 - nt) = o \end{cases}$$

Eliminata deinde quantitate t prodit

$$(3.)\quad bnv^2 - amv_1^2 - abnu + abmu_1 = o$$

commutatisque, ut supra dixi, coordinatis apparet

$$(4.)\quad \left.\begin{matrix} q^2 (bn - am) y^2 - 2pq (bn + am) xy + p^2 (bn - am) x^2 \\ - abp (m + n) y + abq (m - n) x \end{matrix}\right\} = o$$

in qua aequatione p pro Sin. ω et q pro Cos. ω scriptum est.

Ut species curvarum in hanc aequationem quadrantium cognoscatur, quantitates ex eadem noto modo deductae,

$$(5.)\quad 4 p^2 q^2 (bn - am)^2 - 4 p^2 q^2 (bn + am)^2$$

ratio habenda est. Ac primum quidem intelligitur, aequationem (4.) in ellipsin convenire, si quantitatis (5.) valor est positivus, sive

$$\begin{cases} (bn - am)^2 > (bn + am)^2 \\ pq > o \end{cases}$$

Atque si ponitur, quantitates m et n semper esse positivas, quantitates a et b signis diversis affectas esse, sive parabolarum ramos contrarias partes versus jacere oportet. E. g. si ponitur $\frac{b}{a} = -h$, et $m = n$, aeq. (4.) transit in hanc

$$(6.)\quad o = q^2 (h+1) y^2 - 2pq (h-1) xy + p^2 (h+1) x - 2ahpy$$

quae novis coordinatis introductis

$$w = x - \frac{a\,(h-1)}{4q}$$

$$z = y - \frac{ap\,(h+1)}{4q^2}$$

commodiorem induit formam

$$o = q^2 (h+1)\, z^2 - 2pq\,(h-1)\, wz + p^2\,(h+1)\, w^2 - \frac{a^2 p^2 h}{4q^2}\,(h+1) \qquad (7.)$$

Verum ut porro membrum wz secum gerens removeatur, situs quoque axium coordinatarum mutandus est, et angulus γ, quem novi abscissarum axes cum prioribus formant, ejusmodi ab aeq. (7.) pendet, ut sit

$$\text{Sin.}^2 \gamma = \frac{1}{2} - \frac{(h+1)\,(p^2 - q^2)}{2\sqrt{(h+1)^2 (p^2 - q^2)^2 + 4p^2 q^2 (h-1)^2}}$$

$$\text{Cos.}^2 \gamma = \frac{1}{2} + \frac{(h+1)\,(p^2 - q^2)}{2\sqrt{(h+1)^2 (p^2 - q^2)^2 + 4p^2 q^2 (h-1)^2}}$$

Jam si novas coordinatas per r et s denotamus et ponimus

$$z = r\ \text{Sin.}\ \gamma + s\ \text{Cos.}\ \gamma$$

$$w = r\ \text{Cos.}\ \gamma - s\ \text{Sin.}\ \gamma$$

haec evadit aequatio

$$\left.\begin{array}{r} r^2\left[(h+1) + \sqrt{(h+1)^2 (p^2 - q^2)^2 + 4p^2 q^2 (h-1)^2}\right] \\ + s^2\left[(h+1) - \sqrt{(h+1)^2 (p^2 - q^2)^2 + 4p^2 q^2 (h-1)^2}\right] \\ - \dfrac{a^2 p^2 h}{2q^2}\,(h+1) \end{array}\right\} = o \qquad (8.)$$

Quam in ellipsin quadrare, facile intelligitur, quia est

$$h+1 > \sqrt{(h+1)^2 (p^2 - q^2)^2 + 4p^2 q^2 (h-1)^2}$$

Exemplo numerali addito res in clariori luce collocabitur. Ponamus igitur

$$h = 2,\ p^2 = q^2 = \frac{1}{2}$$

tunc aeq. (8.) commutatur in

$$4r^2 + 2s^2 - 3a^2 = o \qquad (9.)$$

Restat, ut ellipseos situs parabolas genetrices versus disquiratur, quarum aequationes nunc sunt

$$\left\{\begin{array}{l} v^2 - au = o \\ v_{\scriptscriptstyle 1}^2 - 2au_{\scriptscriptstyle 1} = o \end{array}\right.$$

et quarum axes angulum rectum includunt, cum sit

$$\text{Sin.}\ 2\omega = 2\ \text{Sin.}\ \omega.\ \text{Cos.}\ \omega = 2pq = 1$$

Ergo parabolae ante motum, ut fig. 1 docet sitae sunt. Parabola NEQ a puncto E in linea EA, altera MEN ab eodem puncto in linea ED profecta cogitanda est. Unde intelligitur, lineas GH et IK, quarum altera angulum AED, altera angulum AEC medium dividit, esse axes coordinatarum in aeq. (4.) addibitarum; sed GH est axis abscissarum x et IK ordinatarum y.

Aeq. (4.) hoc casu formam induit

(10.) $$\frac{3}{2}y^2 - xy + \frac{3}{2}x^2 - 4ay\sqrt{\frac{1}{2}} = o$$

et aeq. (7.) transit in

(11.) $$3z^2 - 2wz + 3w^2 - a^2 = o$$

quarum aequationum coordinatae hanc habent relationem

$$\begin{cases} x = w + \frac{a}{4}\sqrt{2} \\ y = z + \frac{3}{4}a\sqrt{2} \end{cases}$$

Si igitur $ES = \frac{a}{4}\sqrt{2}$, $ER = \frac{3}{4}a\sqrt{2}$ sumitur, erit TU axis ($w = o$), TV axis ($z = o$), et T initium coordinatarum w et z, idemque ellipseos centrum. Quibus exploratis, angulus adhuc, quem axes ellipseos cum axibus coordinatarum includant indagandus est. Qua de causa observemus, esse pro puncto W semiaxis magni TW extremo

$$z^2 + w^2 = \frac{3}{2}a^2$$

quae aequatio conjuncta cum aeq. (11.) edit

$$z - w = o$$

unde elucet, esse

$$\angle VTW = + 45°$$

Pari modo pro puncto Z, in quo semiaxis parvus TZ ellipsi occurrit, invenitur

$$z + w = o$$

$$\text{ergo } \angle VTZ = - 45°$$

Eadem reperiuntur, si valoribus quantitatum $\text{Sin.}^2\,\gamma$ et $\text{Cos.}^2\,\gamma$ antea propositis utimur.

Aequatio (8.) in circulum cadet, si

$$(h+1)^2\,(p^2 - q^2)^2 + 4p^2q^2\,(h-1)^2 = o$$

$$\text{i. e. } (h+1)^2\,\text{Cos.}^2\,2\omega + (h-1)^2\,\text{Sin.}^2\,2\omega = o$$

$$\text{ergo } \text{Sin.}^2\,2\omega = \frac{(h+1)^2}{(h+1)^2 - (h-1)^2}$$

id quod postulat, ut sit $h - 1 = o$, ideoque

$$\text{Sin.}^2\,2\omega = 1$$

$$\text{sive } p^2 = q^2 = \frac{1}{2}$$

Quo facto aeq. (8.) convertitur in hanc circuli aequationem

(12.) $$r^2 + s^2 - \frac{a^2}{2} = o$$

Quae cum ita sint, parabolae, quae circulum gignant, eandem parametrum a habeant, et cum pari celeritate in abscissarum axibus CD et EA fig. (2.) necesse est ita moveantur, ut angulus $AED = 90°$ sit. Locus centri, qui sit, ex aeq. (4.) perspicitur, quae hoc casu formam sumit

(13.) $$y^2 + x^2 - ay\sqrt{2} = o$$

et coordinatas centri edit

$$\begin{cases} x = o \\ y = \frac{a}{2}\sqrt{2} \end{cases}$$

Si ergo $EO = \frac{a}{2}\sqrt{2}$ ponitur, punctum O centrum circuli erit.

Jam vero si ad aeq. (4.) revertimur, hyperbolam proventuram esse videmus, si

$$4p^2q^2\,(bn - am)^2 < 4p^2q^2\,(bn + am)^2$$

$$\text{i. e.} \begin{cases} (bn - am)^2 < (bn + am)^2 \\ p^2q^2 > o \end{cases}$$

unde concluditur, quantitatibus a et b signa eadem esse tribuenda. E. g. si

$$\frac{b}{a} = +h,\ m = n$$

ponimus, aeq. (4.) commutatur in hanc

$$q^2(h-1)y^2 - 2pq(h+1)xy + p^2(h-1)x^2 - 2phay = o \qquad (14.)$$

et substitutis valoribus

$$x = w - \frac{a(h+1)}{4q}$$

$$y = z - \frac{ap(h-1)}{4q^2}$$

in hanc

$$q^2(h-1)z^2 - 2pq(h+1)wz + p^2(h-1)w^2 + \frac{p^2}{4q^2}h.(h-1)a^2 = o \qquad (15.)$$

Ex qua eodem modo, atque supra, elicitur

$$\left.\begin{array}{l} r^2\,[(h-1) + \sqrt{(h-1)^2(p^2-q^2)^2 + 4p^2q^2(h+1)^2}] \\ -s^2\,[(h-1) - \sqrt{(h-1)^2(p^2-q^2)^2 + 4p^2q^2(h+1)^2}] \\ + \frac{p^2}{2q^2}.h\,(h-1)\,a^2 \end{array}\right\} = o \qquad (16.)$$

quam in hyperbolam quadrare quisque intelligit. Jam ut res exemplo numerali illustretur, ponere lubet

$$h = 2,\ p^2 = q^2 = \frac{1}{2}$$

quo facto aeq. (16.) convertitur in

$$4r^2 - 2s^2 + a^2 = o \qquad (17.)$$

Fig. (3.) hyperbolae huic aequationi satisfacientis dimidia pars HSG, et parabolae genetrices LEG et GEK delineatae sunt.

Parabolae ex aeq. (4.) prodeunt, si

$$4p^2q^2\,[(bm - an)^2 - (bm + an)^2] = o,$$

id quod, nisi est $p = o$, aut $q = o$, cogitari non potest. Verum si ponitur e. g. $p = o$, aequatio parabolae efficitur

$$(18.) \qquad y^2 (bm - an) + ab (n - m) x = o$$

atque si $q = o$

$$(19.) \qquad x^2 (bn - am) - ab (m + n) y = o$$

Denique parabolarum sectiones in linea recta sibi succedunt, si

$$(20.) \quad q^4(bn-am)(m-n)^2a^2b^2-2p^2q^2(bn+am)(m^2-n^2)a^2b^2+p^4(bn-am)(m+n)^2a^2b^2=o$$

quae conditio, aequatione (4) collata cum aequatione generali

$$Ay^2 + Bxy + Cx^2 + Dx + Ey = o$$

mox reperitur, nam haec aequatio in lineam rectam quadrat

$$AE^2 - BDE + CD^2 = o$$

Jam si ponimus $\frac{b}{a} = h$, $\frac{m}{n} = k$, ex aeq. (20.) prodit

$$(21.) \qquad q^4 (h-k)(h-1)^2 - 2p^2 q^2 (h+k)(h^2-1) + p^4 (h-k)(h+1)^2 = o$$

unde deducitur

$$(22.) \qquad \frac{p^2}{q^2} = \frac{k-1}{(k+1)(h-k)} [h + k \pm 2\sqrt{hk}]$$

itaque poscitur, ut h sit positivum, quia k quoque positivum, atque $(k - 1)$ et $(h - k)$ signis inter se congruant. Ex aeq. (22.) sequitur porro

$$p^2 = \frac{(k-1)(\sqrt{h} \pm \sqrt{k})^2}{(k-1)(\sqrt{h} \pm \sqrt{k})^2 + (k+1)(h-k)}$$

$$q^2 = \frac{(k+1)(h-k)}{(k-1)(\sqrt{h} \pm \sqrt{k})^2 + (k+1)(h-k)},$$

quibus valoribus in aeq. (4.) substitutis evadit aequatio

$$o = \begin{cases} M(h-k)y^2 - 2(h+k)\sqrt{MN}\,.\,xy + N(h-k)x^2 \\ -(k+1)ah\sqrt{(MN+N^2)}\,.\,y + (k-1)ah\sqrt{(MN+M^2)}\,.\,x \end{cases}$$

in qua M pro $(k+1)(h-k)$ et N pro $(k-1)(\sqrt{h} \pm \sqrt{k})^2$ positum est. Jam si ponitur

$$x = u - \frac{a(k+1)}{4kM}\sqrt{hkM(M+N)}$$

$$y = v - \frac{a(k-1)}{4kN}\sqrt{hkN(M+N)},$$

inde efficitur

$$M(h-k)v^2 - 2(h+k)\sqrt{MN}\,.\,uv + N(h-k)u^2 = o$$

sive

$$o = \begin{cases} (k+1)(h-k)^2 v^2 \mp 2(h+k)(\sqrt{h} \pm \sqrt{k})\sqrt{(k^2-1)(h-k)} \\ + (h-k)(k-1)(\sqrt{h} \pm \sqrt{k})^2 u^2 \end{cases}$$

unde deducitur aequatio

$$\frac{v}{u} = \frac{(\sqrt{h} \pm \sqrt{k})^3}{(k+1)(h-k)^2}\sqrt{(k^2-1)(h-k)}$$

quae in duas lineas rectas quadrat. Sed una tantum re vera nata est, nam signum quantitatis $\sqrt{k}$ ab ejusdem quantitatis signo in aeq. (22.) pendet. Ut nil moremur, exemplis numeralibus praetermissis, ad curvas transeamus, quae aliis secundi gradus curvis gignantur.

Aequationes duarum ellipsium

$$\left\{\begin{array}{l} u^2 + a^2 v^2 = b^2 \\ u_1^2 + c^2 v_1^2 = d^2 \end{array}\right. \tag{23.}$$

modo jam dicto transeunt in has

$$\left\{\begin{array}{l} (u - mt)^2 + a^2 v^2 = b^2 \\ (u_1 - nt)^2 + c^2 v_1^2 = d^2 \end{array}\right. \tag{24.}$$

quae eliminata quantitate t coalescunt in

$$nu - mu_1 = n\sqrt{b^2 - a^2 v^2} - m\sqrt{d^2 - c^2 v_1^2} \tag{25.}$$

unde novis coordinatis ratione memorata introductis nascitur

$$\left.\begin{array}{l} x^4 [q^4(m-n)^4 + 2p^2q^2(m-n)^2(a^2n^2+c^2m^2) + p^4(a^2n^2-c^2m^2)^2] \\ + y^4 [p^4(m+n)^2 + 2p^2q^2(m+n)^2(a^2n^2+c^2m^2) + q^4(a^2n^2-c^2m^2)^2] \\ + 2x^2y^2 \{ p^2q^2[(m^2-n^2)^2 + (a^2n^2+c^2m^2)^2 + 4m^2n^2a^2c^2 + 2[n^2(1-a^2) - m^2(1-c^2)]^2] \\ \qquad + (a^2n^2+c^2m^2)[p^4(m+n)^2 + q^4(m-n)^2] \} \\ + 4pqx^3y[n^2(1-a^2) - m^2(1-c^2)][q^2(m-n)^2 + p^2(a^2n^2+c^2m^2)] \\ + 4pqxy^3[n^2(1-a^2) - m^2(1-c^2)][p^2(m+n)^2 + q^2(a^2n^2+c^2m^2)] \\ - 2x^2 \{ (n^2b^2+m^2d^2)[q^2(m-n)^2 + p^2(a^2n^2+c^2m^2)] - 2m^2n^2p^2(a^2d^2+b^2c^2) \} \\ - 2y^2 \{ (n^2b^2+m^2d^2)[p^2(m+n)^2 + q^2(a^2n^2+c^2m^2)] - 2m^2n^2q^2(a^2d^2+b^2c^2) \} \\ - 4pqxy \{ (n^2b^2+m^2d^2)[n^2(1-a^2) - m^2(1-c^2)] + 2m^2n^2(a^2d^2-b^2c^2) \} \\ + (n^2b^2 - m^2d^2)^2 \end{array}\right\} = o \tag{26.}$$

Quum vero de natura curvarum in hanc aequationem cadentium in universum parum dici possit, casus singulares adumbrare volumus. Hoc animo conditiones disquirantur, quibus efficiatur, ut membra x^3y et xy^3 evanescant. Sunt

$$\text{I} \quad p = o\,; \quad \text{II} \quad q = o\,; \quad \text{III} \quad n^2(1 - a^2) = m^2(1 - c^2);$$

$$\text{IV} \quad \left\{\begin{array}{l} (m - n)^2 q^2 + (a^2n^2 + c^2m^2) p^2 = o \\ (m + n)^2 p^2 + (a^2n^2 + c^2m^2) q^2 = o \end{array}\right.$$

quarum ultima, nullum sensum praebens, negligenda est. Sed priores duas, cum efficiant, ut non solum x^3y et xy^3 sed etiam xy removeantur, primum illustrare in animo est. Si ponitur $p = o$ aeq. (26.) abit in hanc

$$o = \left\{\begin{array}{l} (m-n)^4 x^4 + (a^2n^2 - c^2m^2)^2 y^4 + 2(m-n)^2(a^2n^2+c^2m^2)x^2y^2 \\ - 2(m-n)^2(n^2b^2+m^2d^2)x^2 - 2(a^2n^2 - m^2c^2)(n^2b^2 - m^2d^2)y^2 \\ + (n^2b^2 - m^2d^2)^2 \end{array}\right. \tag{27.}$$

ex qua statim apparet, omnes curvas huc quadrantes, cum membrum summum in factores simplices secerni non possit, clausas esse. Atque porro mox intelligitur, curvas ab utroque coordinatarum axe in aequales partes diduci. Attamen quoniam axis $x = o$ curvarum ramis occurrat omnino non necesse est, quia pro $x = o$ evadit

$$y = \pm \sqrt{\frac{n^2b^2 - m^2d^2}{a^2n^2 - m^2c^2}} \tag{28.}$$

abscissarum vero axis semper secatur, quia pro $y = o$ est

$$(29.) \qquad x = \pm \frac{nb \pm md}{m - n}$$

concludere itaque licet, curvas in universum duas partes clausas continere, quarum quaeque ab abscissarum axe in medio secatur.

Ex aeq. (28.) cognoscitur, axem ordinatarum a curvis secari, si valor quantitatis

$$\frac{n^2 b^2 - m^2 d^2}{a^2 n^2 - m^2 c^2}$$

positivus est, id quod efficitur per

$$(\alpha.) \begin{cases} \frac{b}{d} > \frac{m}{n} \\ \frac{a}{c} > \frac{m}{n} \end{cases} \quad \text{aut} \quad (\beta.) \begin{cases} \frac{b}{d} < \frac{m}{n} \\ \frac{a}{c} < \frac{m}{n} \end{cases}$$

Neque hae tamen sufficiunt conditiones, aliae adhuc investigandae sunt. Ex aeq. (27.) evadit

$$(30.) \quad \begin{aligned} (a^2n^2 - c^2m^2)^2 y^2 = &-(m-n)^2(a^2n^2 + c^2m^2)x^2 + (a^2n^2 - c^2m^2)(n^2b^2 - m^2d^2) \\ &\pm 2mn(m-n)x\sqrt{a^2c^2(m-n)^2x^2 + (a^2n^2 - c^2m^2)(a^2d^2 - b^2c^2)} \end{aligned}$$

ergo postulatur, ut cum conditione (α.) haec nova

$$(\alpha'.) \quad \frac{a}{c} > \frac{b}{d} \text{ aut } \frac{a}{c} = \frac{b}{d}$$

atque cum (β.) haec altera

$$(\beta'.) \quad \frac{a}{c} < \frac{b}{d} \text{ aut } \frac{a}{c} = \frac{b}{d}$$

conjuncta sit; alias enim curvae pro

$$x < \pm \sqrt{-\frac{(a^2n^2 - c^2m^2)(a^2d^2 - b^2c^2)}{a^2c^2(m-n)^2}}$$

non existunt, et coordinatae

$$x = o$$

$$y = \pm \sqrt{\frac{n^2b^2 - m^2d^2}{a^2n^2 - c^2m^2}}$$

puncta discreta denotant.

Neque negligendum est, curvas axi $x = o$ tunc quoque occurrere, si pro $x = o$ invenitur $y = o$, id quod evenit si

$$\frac{b}{d} = \frac{m}{n}$$

quo in casu aeq. (27.) convertitur in hanc

$$(31.) \quad (m-n)^4x^4 + (a^2n^2 - c^2m^2)^2y^4 + 2(m-n)^2(a^2n^2 + c^2m^2)x^2y^2 - 2(m-n)^2m^2d^2x^2 = o$$

sive

$$(32.) \quad (a^2n^2 - c^2m^2)^2y^2 = -(m-n)^2(a^2n^2 + c^2m^2)x^2 \pm 2mn(m-n)x\sqrt{a^2c^2(m-n)^2x^2 + \frac{d^2}{n^2}(a^2n^2 - c^2m^2)^2}$$

Unde cum deducatur

$$\frac{dy}{dx}=\frac{2mn(m-n)^3a^2c^2x^2+\frac{m}{n}(m-n)d^2(a^2n^2-c^2m^2)^2\mp(m-n)^2(a^2n^2+c^2m^2)x\sqrt{a^2c^2(m-n)^2x^2+\frac{d^2}{n^2}(a^2n^2-c^2m^2)^2}}{(a^2n^2-c^2m^2)^2y\sqrt{a^2c^2(m-n)^2x^2+\frac{d^2}{n^2}(a^2n^2-c^2m^2)^2}}$$

itaque hoc casu axem $x=o$ in puncto $y=o$ utrimque tangi, quisque intelligit. Quae cum ita sint, curvarum partes sese neque tangunt, neque secant, si conditionum

$$(\gamma.)\begin{cases}\frac{b}{d}>\frac{m}{n}\\ \frac{a}{c}<\frac{m}{n}\end{cases}\qquad(\delta.)\begin{cases}\frac{b}{d}<\frac{m}{n}\\ \frac{a}{c}>\frac{m}{n}\end{cases}$$

$$(\varepsilon.)\begin{cases}\frac{b}{d}>\frac{m}{n}\\ \frac{a}{c}>\frac{m}{n}\\ \frac{a}{c}<\frac{b}{d}\end{cases}\qquad(\zeta.)\begin{cases}\frac{b}{d}<\frac{m}{n}\\ \frac{a}{c}<\frac{m}{n}\\ \frac{a}{c}>\frac{b}{d}\end{cases}$$

una existit. Adformam curvarum accuratius cognoscendam, contemplemur coefficientem differentialem aequationis (30.) $\frac{dy}{dx}=$

$$\frac{\ldots-n)\left[2mn(m-n)^2a^2c^2x^2+mn(a^2n^2-c^2m^2)(a^2d^2-b^2c^2)\mp(m-n)(a^2n^2+c^2m^2)\sqrt{a^2c^2(m-n)^2x^2+(a^2n^2-c^2m^2)(a^2d^2-b^2c^2)}\right]}{(a^2n^2-c^2m^2)^2y\sqrt{a^2c^2(m-n)^2x^2+(a^2n^2-c^2m^2)(a^2d^2-b^2c^2)}} \quad (33)$$

Primum, ut efficiatur $\frac{dy}{dx}=o$, numeratore $=o$ posito, pro abscissa x valores prodeunt

$$x^2=-\frac{(a^2d^2-b^2c^2)n^2}{c^2(m-n)^2}$$

$$x^2=+\frac{(a^2d^2-b^2c^2)m^2}{a^2(m-n)^2},$$

quorum alter semper, prout $\frac{a}{c}>\frac{b}{d}$ aut $\frac{a}{c}<\frac{b}{d}$, cogitari non potest. In illum quadrat

$$y=\pm\frac{d}{c}$$

et in hunc

$$y=\pm\frac{b}{a}$$

Ergo quatuor tantum exstant puncta in quibus tangentes et axis abscissarum paralleli sunt. Porro cernimus, coefficientem diff. valorem infinitum habere pro

$$y=o$$

aut $a^2c^2(m-n)^2x^2+(a^2n^2-c^2m^2)(a^2d^2-b^2c^2)=o$

quarum conditionum prior in omnes aequationis (27.) curvas convenit, altera autem cogitari non potest, nisi exstat conditio ($\varepsilon.$) aut ($\zeta.$). Verum cum pro

$$x=\pm\sqrt{-\frac{(a^2n^2-c^2m^2)(a^2d^2-b^2c^2)}{a^2c^2(m-n)^2}} \qquad (34.)$$

evadat

$$(35.)\qquad y = \pm\sqrt{\frac{a^4n^2d^2 - c^4b^2m^2}{a^2c^2(a^2n^2 - c^2m^2)}}$$

ergo insuper poscitur, ut conditio (ε.) conjuncta sit cum hac nova

$$(\eta.)\quad \frac{a^2}{c^2} > \frac{mb}{nd}$$

et conditio (ζ.) cum hac altera

$$(\vartheta.)\quad \frac{a^2}{c^2} < \frac{mb}{nd}$$

Itaque, cum in universum quatuor tantum puncta cogitari possint, in quibus tangentes ad axem abscissarum perpendiculares sint, curvae duas ovales partes continent. Verum si (ε.) cum (η.) aut (ζ.) cum (ϑ.) conjunctum exstat, alia adhuc puncta quatuor apparent, in quibus $\frac{dy}{dx} = \infty$. Atque cum, coordinatas eorum

$$\begin{cases} x' = \pm\sqrt{-\dfrac{(a^2n^2 - c^2m^2)(a^2d^2 - b^2c^2)}{a^2c^2(m-n)^2}} \\ y' = \pm\sqrt{\dfrac{a^4n^2d^2 - c^4b^2m^2}{a^2c^2(a^2n^2 - c^2m^2)}} \end{cases}$$

cum coordinatis jam antea dictis

$$\begin{cases} x'' = \pm\dfrac{nb \pm md}{m-n} \\ y'' = o \end{cases}$$

comparantes inveniamus, semper esse $x' < x''$, concludere est justum, curvas hoc in casu cardiformes esse. Quae dixi, ut exemplis illustrem, primo ponere juvat $\frac{a}{c} = \frac{b}{d}$, quod una conditioni ($\alpha$.) et ($\beta$.) satisfacit. Inde aeq. (27.) commutatur in hanc

$$(36.)\quad o = \begin{cases} (m-n)^4x^4 + (a^2n^2 - c^2m^2)^2y^4 + 2(m-n)^2(a^2n^2 + c^2m^2)x^2y^2 \\ - 2\dfrac{d^2}{c^2}(m-n)^2(a^2n^2 + m^2c^2)x^2 - 2\dfrac{d^2}{c^2}(a^2n^2 - c^2m^2)^2y^2 \\ + \dfrac{d^4}{c^4}(a^2n^2 - c^2m^2)^2 \end{cases}$$

sive

$$(37.)\quad o = \left[(m-n)^2x^2 + (an+cm)^2y^2 - \frac{d^2}{c^2}(an+cm)^2\right]\left[(m-n)^2x^2 + (an-cm)^2y^2 - \frac{d^2}{c^2}(an-cm)^2\right]$$

quam in duas ellipses, quae in punctis

$$\begin{cases} x = o \\ y = \pm d \end{cases}$$

sese tangunt, convenire, quisque videt. Sit e. g. $a = 2$, $c = 1$, $b = 2d$, $n = 2m$, inde nascitur

$$(38.)\qquad x^4 + 225y^4 + 34x^2y^2 - 34d^2x^2 - 450d^2y^2 + 225d^4 = o$$

i. e. $\qquad (x^2 + 25y^2 - 25d^2)(x^2 + 9y^2 - 9d^2) = o,$

quibus duobus factoribus ellipses duae fig. (4.) adumbratae satisfaciunt.

Jam ad alios, conditionem $\frac{a}{c} = \frac{b}{d}$ rejicientes, casus ut transeamus, primo ponamus

$$\frac{a}{c} = 1; \; \frac{b}{d} = \frac{1}{\sqrt{2}}; \; \frac{m}{n} = \frac{1}{2},$$

ita ut conditio (α.) cum (α') conjuncta appareat. Hac ratione cum nascatur

$$x^4 + 9y^4 + 10x^2y^2 - 12b^2y^2 - 12b^2x^2 + 4b^4 = o \qquad (39.)$$

sive

$$9y^2 = -5x^2 + 6b^2 \pm 4x\sqrt{3b^2 + x^2}$$

statim animadvertimus, curvam abscissarum axi occurrere in punctis

$$x = \pm (2 \pm \sqrt{2})\, b$$

atque axi ordinatarum in his

$$y = \pm b\sqrt{\frac{2}{3}}.$$

In illis simul tangens ad axem abscissarum perpendicularis est, id quod pro axe ordinatarum in his evenit punctis

$$\begin{cases} x = \pm b \\ y = \pm b \end{cases}$$

Quae cum ita sint, curvam fig. (5.) adumbratam esse, elucet. Idem quum conditionibus (β.) et (β') conjunctis efficiatur, ad exemplum conditioni $\frac{b}{d} = \frac{m}{n}$ respondens transire licet. Hoc consilio si ponimus

$$\frac{a}{c} = 1, \; \frac{b}{d} = \frac{m}{n} = 2$$

prodit aequatio

$$x^4 + 9y^4 + 10x^2y^2 - 16d^2x^2 = o \qquad (40.)$$

sive

$$9y^2 = -5x^2 \pm 4x\sqrt{9d^2 + x^2}$$

in curvam quadrans, quae axi abscissarum in punctis

$$x = o, \quad \text{et} \quad x = \pm 4d$$

occurrit, axem vero ordinatarum in puncto

$$(x = o; \; y = o)$$

ab utraque parte tangit. Cf. fig. (6.)

Restat adhuc, ut casus illustrentur, in quibus curvarum partes inter se separatae sunt. Quamobrem, ut conditioni (γ.) satisfiat, ponentes

$$\frac{a}{c} = 1, \; \frac{m}{n} = 2, \; \frac{b}{d} = 3$$

aequationi occurrimus

$$x^4 + 9y^4 + 10x^2y^2 + 30d^2y^2 - 26d^2x^2 + 25d^4 = o \qquad (41.)$$

sive

$$9y^2 = -5(x^2 + 3d^2) \pm 4x\sqrt{x^2 + 24d^2}$$

unde, cum pro $x = o$ evadat $9y^2 = -15d^2$, cognoscimus, curvarum partes re vera sibi occurrere non posse. Cf. fig. (7.)

Conditiones (δ.), (ε.) et (ζ.) idem efficientes exemplis illustrare, quum necesse non habeamus, exemplum in conditionem (ε.) cum (η.) conjunctam cadens edere volumus. Hac mente si ponimus

$$\frac{a}{c} = 1, \quad \frac{m}{n} = \frac{1}{3}, \quad \frac{b}{d} = 2$$

provenit aequatio

$$(42.) \qquad 64y^4 + 80x^2y^2 + 16x^4 - 560d^2y^2 - 296d^2x^2 + 1225d^4 = o$$

sive

$$8y^2 = -5x^2 + 35d^2 \pm 3x\sqrt{x^2 - 6d^2}$$

ex qua primo adspectu cognoscimus, curvam inter

$$x = -d\sqrt{6} \text{ et } x = +d\sqrt{6}$$

existere non posse; quum tamen pro $x = o$ sit

$$y = \pm d\sqrt{\frac{35}{8}},$$

ergo duo exstant puncta discreta. Porro progressum ordinatarum spectantes, mox videmus, in quamque abscissam inter

$$x = \pm d\sqrt{6} \text{ et } x = \pm \frac{5}{2} d$$

quatuor reales ordinatas cadere, atque inter

$$x = \pm \frac{5}{2} d \text{ et } x = \pm \frac{7}{2} d,$$

quibus in punctis curva axi abscissarum occurrit, duas tantum. Ad curvae formam accuratius cognoscendam si intuemur

$$\frac{dy}{dx} = \frac{3(x^2 - 3d^2) \mp 5x\sqrt{x^2 - 6d^2}}{8y\sqrt{x^2 - 6d^2}},$$

mox intelligimus, tangentes ad axem abscissarum perpendiculares esse non solum in verticibus curvae sed etiam in punctis

$$\begin{cases} x = \pm d\sqrt{6} \\ y = \pm \frac{d}{4}\sqrt{10} \end{cases}$$

Tum vero ratione habita coefficientis diff. secundi

$$\frac{d^2y}{dx^2} = \frac{15d^2 . x^2(10x^2 - 63d^2) \pm d^2 . (202x^2 - 969d^2)\sqrt{x^2 - 6d^2}}{64y^3\sqrt{(x^2 - 6d^2)^3}},$$

quippe qui evanescat pro

$$\begin{cases} x = \pm 2,4676\,d \\ y = \pm 0,541\,d \end{cases}$$

concludere possumus, curvam his in punctis inter $x = \pm d\sqrt{6}$ et $x = \pm \frac{5}{2} d$ sitis inflexionem habere. Cf. fig. 8, in qua dimidia pars delineata est.

Jam ad conditionem II $q=o$ transeuntes ad aequationem pervenimus

$$\left.\begin{array}{r}(a^2n^2-c^2m^2)^2x^4+(m+n)^4y^4+2(m+n)^2(a^2n^2+c^2m^2)x^2y^2\\ -2(a^2n^2-c^2m^2)(n^2b^2-m^2d^2)x^2-2(m+n)^2(n^2b^2+m^2d^2)y^2\\ +(n^2b^2-m^2d)^2\end{array}\right\}=o \qquad (43.)$$

quae, quum ad curvas curvis jam adumbratis similes ducat, praetermittenda videtur. Itaque adhuc conditio III

$$n^2(1-a^2)=m^2(1-c^2)$$

illustranda est, ex qua nascitur

$$o=\left\{\begin{array}{l}[(m-n)^2x^4+(m+n)^2y^4+2x^2y^2(m^2-n^2+2a^2n^2)][m-n)^2q^4+(m+n)^2p^4+2p^2q^2(m^2-n^2+2a^2n^2)]\\ -2x^2\left\{(n^2b^2+m^2d^2)[q^2(m-n)^2+p^2(m^2-n^2+2a^2n^2)]-2m^2n^2p^2[a^2d^2+\frac{b^2}{m^2}(m^2-n^2+a^2n^2)]\right\}\\ -2y^2\left\{(n^2b^2+m^2d^2)[p^2(m+n)^2+q^2(m^2-n^2+2a^2n^2)]-2m^2n^2q^2[a^2d^2+\frac{b^2}{m^2}(m^2-n^2+a^2n^2)]\right\}\\ -8xypqm^2n^2\left[a^2d^2-\frac{b^2}{m^2}(m^2-n^2+a^2n^2)\right]\\ +(n^2b^2-m^2d^2)^2\end{array}\right. \qquad (44)$$

Omnes huc cadentes curvas clausas esse oportet, nam membrum summum, quum coefficiens quantitatis x^2y^2

$$m^2-n^2+2a^2n^2=a^2n^2+c^2m^2$$

semper positivus sit, in factores simplices secerni non potest. Si e. g. ponimus

$$a=c=1;\ b=1,\ d=2;\ m=n=1;\ p^2=q^2=\frac{1}{2}$$

prodit aequatio

$$8y^4+8x^2y^2-20y^2-12xy+9=0 \qquad (45.)$$

sive

$$x=\frac{1}{y}\left[\frac{3}{4}\pm\sqrt{\left(\frac{9}{4}-y^2\right)\left(y^2-\frac{1}{4}\right)}\right]$$

quam intuiti mox cognoscimus, curvae ramos finitos tantum inter

$$y'=+\frac{1}{2} \text{ et } y''=+\frac{3}{2}$$

atque ab altera parte inter

$$y'''=-\frac{1}{2} \text{ et } y^{\text{iv}}=-\frac{3}{2}$$

exstare. Abscissae in has ordinatas convenientes sunt

$$x'=+\frac{3}{2},\quad x''=+\frac{1}{2}$$

$$x'''=-\frac{3}{2},\quad x^{\text{iv}}=-\frac{1}{2},$$

quibus coordinatarum valoribus simul efficitur, ut sit $\frac{dy}{dx}=o$, nam est

$$\frac{dy}{dx}=\frac{y(4xy-3)}{8y^3+4x^2y-10y-3x}.$$

Sed ad abscissarum fines indagandos inter denominatorem quantitatis $\frac{dy}{dx}$ et aequationem (45.) x eliminantes novam invenimus aequationem

$$128y^8 - 72y^4 - 180y^2 + 81 = o$$

cujus radices reales quatuor sunt

$$y' = +0{,}635;\quad y'' = +1{,}069;\quad y''' = -0{,}635;\quad y^{IV} = -1{,}069;$$

unde evadunt abscissae quaesitae

$$x' = +2{,}012;\quad x'' = -0{,}23;\quad x''' = -2{,}012;\quad x^{IV} = +0{,}23$$

Abscissarum axem a curva non secari, jam vidimus, sed ordinatarum axis curvae quatuor in punctis occurrit, quae ita denotantur

$$y = \pm \frac{1}{2}\sqrt{5 \pm \sqrt{7}}$$

i. e. $\quad y = \pm 1{,}3826$, et $y = \pm 0{,}7671$.

Quae cum ita sint, curvam duas congruentes et clausas partes continere, judicandum est. Cf. fig. 9, in qua dimidia curva adumbrata est.

Postremo negligendum non est, aequationem (44.), si ponatur

$$a^2d^2 - \frac{b^2}{m^2}(m^2 - n^2 + a^2n^2) = o,$$

in duas alias aequationes gradus secundi secerni posse. Hac enim lege est

$$(46.)\quad \left.\begin{aligned}&[x^4(m-n)^2+y^4(m+n)^2+2x^2y^2(m^2-n^2+2a^2n^2)][q^4(m-n)^2+2p^2q^2(m^2-n^2+2a^2n^2)+p^4(m+n)^2]\\&\quad -2x^2\frac{b^2}{a^2}(m-n)^2[p^2(m+n)^2+q^2(m-n^2+2a^2n^2)]\\&\quad -2y^2\frac{b^2}{a^2}(m+n)^2[q^2(m-n)^2+p^2(m^2-n^2+2a^2n^2)]\\&\quad +\frac{b^4}{a^4}(m^2-n^2)^2\end{aligned}\right\} = o$$

unde prodeunt duae aequationes

$$(47.)\quad o = \left\{\begin{aligned}&[y^2(m+n)^2+x^2(an+\sqrt{m^2-n^2+a^2n^2})^2][(n-m-2np^2)^2+4p^2q^2a^2n^2]\\&-\frac{b^2}{a^2}(m+n)^2[q^2(m-n)^2+p^2(an+\sqrt{m^2-n^2+a^2n^2})^2]\end{aligned}\right.$$

$$(48.)\quad o = \left\{\begin{aligned}&[y^2(m+n)^2+x^2(an-\sqrt{m^2-n^2+a^2n^2})^2][(n-m-2np^2)^2+4p^2q^2a^2n^2]\\&-\frac{b^2}{a^2}(m+n)^2[q^2(m-n)^2+p^2(an-\sqrt{m^2-n^2+a^2n^2})^2]\end{aligned}\right.$$

quarum quaeque in ellipsin convenit. Si e. g. ponimus $a = 1$, conditionibus dictis

$$n^2(1-a^2) - m^2(1-c^2) = o$$

$$a^2d^2 - \frac{b^2}{m^2}(m^2 - n^2 + a^2n^2) = o$$

efficitur $c = 1$, $b = d$, ita ut aequationes (23.) duos congruentes circulos denotent. Igitur ex aeq. (44.) evadit

$$o = \left\{ \begin{array}{l} [x^4(m-n)^2+y^4(m+n)^2+2x^2y^2(m^2+n^2)][(m-n)^2+4mnp^2] \\ -2x^2b^2(m-n)^2[m^2+n^2+2mnp^2] \\ -2y^2b^2(m+n)^2[m^2+n^2-2mnq^2] \\ +b^4(m^2-n^2)^2 \end{array} \right. \qquad (49.)$$

unde sequitur

$$\left\{ \begin{array}{l} o=y^2+x^2-b^2 \\ o=[y^2(m+n)^2+x^2(m-n)^2][(m-n)^2+4mnp^2]-b^2(m^2-n^2)^2 \end{array} \right. \qquad (50.)$$

Hac ratione cum de conditionibus I, II et III satis copiosa scripsisse existimemus, ad aeq. (26.) revertimur. Verum cum membrum ejus summum formam habeat

$$Ax^4+By^4+Cx^2y^2+Dxy^3+Ex^3y,$$

ergo in universum censeri non possit, quibus in casibus rami infiniti existant, de curvarum forma in universum nihil constituendum est. Quam obrem ad curvas transeamus, quae hyperbolis gignuntur.

Aequationes duarum hyperbolarum

$$\left\{ \begin{array}{l} u^2-a^2v^2=b^2 \\ u_1^2-c^2v_1^2=d^2 \end{array} \right. \qquad (51.)$$

ratione supra commemorata in hanc ducunt aequationem

$$\left. \begin{array}{l} x^4[q^4(m-n)^4-2p^2q^2(m-n)^2(a^2n^2+c^2m^2)+p^4(c^2m^2-a^2n^2)^2] \\ +y^4[p^4(m+n)^4-2p^2q^2(m+n)^2(a^2n^2+c^2m^2)+q^4(c^2m^2-a^2n^2)^2] \\ +2x^2y^2\left\{ p^2q^2[(m^2-n^2)^2+(a^2n^2+c^2m^2)^2+4m^2n^2a^2c^2+2[n^2(1+a^2)-m^2(1+c^2)]] \right. \\ \qquad \left. -(a^2n^2+c^2m^2)[p^4(m+n)^2+q^4(m-n)^2] \right\} \\ +4pqx^3y[n^2(1+a^2)-m^2(1+c^2)][q^2(m-n)^2-p^2(a^2n^2+c^2m^2)] \\ +4pqxy^3[n^2(1+a^2)-m^2(1+c^2)][p^2(m+n)^2-q^2(a^2n^2+c^2m^2)] \\ -2x^2\left\{(n^2b^2+m^2d^2)[q^2(n-m)^2-p^2(a^2n^2+c^2m^2)]+2m^2n^2p^2(a^2d^2+b^2c^2)\right\} \\ -2y^2\left\{(n^2b^2+m^2d^2)[p^2(m+n)^2-q^2(a^2n^2+c^2m^2)]+2m^2n^2q^2(a^2d^2+b^2c^2)\right\} \\ -4pqxy\left\{(n^2b^2+m^2d^2)[n^2(1+a^2)-m^2(1+c^2)]-2m^2n^2(a^2d^2-b^2c^2)\right\} \\ +(n^2b^2-m^2d^2)^2 \end{array} \right\} = o \qquad (52.)$$

Jam primum disquirentes, qua ratione efficiatur, ut membra x^3y et xy^3 evanescant, conditiones invenimus quatuor

$$\text{I}\ p=o;\quad \text{II}\ q=o;\quad \text{III}\ n^2(1+a^2)-m^2(1+c^2)=o$$

$$\text{IV}\ \left\{ \begin{array}{l} q^2(m-n)^2-p^2(a^2n^2+c^2m^2)=o \\ p^2(m+n)^2-q^2(a^2n^2+c^2m^2)=o \end{array} \right.$$

quarum, cum ultima exstare non possit, solae tres priores illustrandae sunt. Quae cum ita sint, primum $p=o$ ponamus, unde efficitur aequatio

$$(53.)\quad o = \begin{cases} (m-n)^4 x^4 + (c^2m^2 - a^2n^2)^2 y^4 - 2(m-n)^2(a^2n^2 + c^2m^2)x^2y^2 \\ -2(n^2b^2 + m^2d^2)(n-m)^2x^2 - 2(c^2m^2 - a^2n^2)(n^2b^2 - m^2d^2)y^2 \\ +(n^2b^2 - m^2d^2)^2 \end{cases}$$

sive

$$(54.)\quad (c^2m^2 - a^2n^2)^2y^2 = \begin{cases} (m-n)^2(a^2n^2 + c^2m^2)x^2 + (c^2m^2 - a^2n^2)(n^2b^2 - m^2d^2) \\ \pm 2mn(m-n)x\sqrt{a^2c^2(m-n)^2x^2 + (c^2m^2 - a^2n^2)(b^2c^2 - a^2d^2)} \end{cases}$$

Curvae huc quadrantes ramos habent infinitos octo, quorum asymptotae rectilineae quantitate

$$(m-n)^4x^4 - 2(m-n)^2(a^2n^2 + c^2m^2)x^2y^2 + (c^2m^2 - a^2n^2)^2y^4 = o$$

tali modo denotantur

$$(55.)\quad \begin{cases} (m-n)x \mp (an + cm)y = o \\ (m-n)x \mp (cm - an)y = o \end{cases}$$

Axis abscissarum ($y = o$) a curvarum ramis semper secatur quatuor in punctis

$$(56.)\quad x = \pm \frac{nb \pm md}{m-n}$$

axis vero ordinatarum, quum sit pro $x = o$

$$(57.)\quad y = \pm\sqrt{\frac{n^2b^2 - m^2d^2}{c^2m^2 - a^2n^2}},$$

duobus in punctis, si quidem apparet

$$\text{aut}\quad (\alpha.)\begin{cases} \frac{b}{d} > \frac{m}{n} \\ \frac{a}{c} < \frac{m}{n} \end{cases}$$

$$\text{aut}\quad (\beta.)\begin{cases} \frac{b}{d} < \frac{m}{n} \\ \frac{a}{c} > \frac{m}{n} \end{cases}$$

Atque hae conditiones solae sufficiunt. Quum enim pro (α.) simul $\frac{b}{d} > \frac{a}{c}$ et pro (β.) simul $\frac{b}{d} < \frac{a}{c}$ prodeat, ita ut quantitas

$$(c^2m^2 - a^2n^2)(b^2c^2 - a^2d^2)$$

in aeq. (54.) sub radice posita negativum valorem induere nequeat, aequatione (57.) puncta discreta nullo modo denotantur. Si e. g. ponitur $a = c = 1$, $b = 3d$, $m = 2n$, inde nascitur

$$(58.)\quad x^4 + 9y^4 - 10x^2y^2 - 26d^2x^2 - 30d^2y^2 + 25d^4 = o$$

Hoc casu curva axi ordinatarum in duobus

$$y = \pm d\sqrt{\frac{5}{3}},$$

axi vero abscissarum in quatuor

$$x = \pm 5d,\quad x = \pm d$$

occurrit punctis. Asymptotarum aequationes sunt

$$\left\{ \begin{array}{l} x \mp 3y = o \\ x \mp y = o \end{array} \right. \tag{59.}$$

Cf. fig. 10.

Verum si apparet $\frac{b}{d} = \frac{m}{n}$, pro $x = o$ simul fit $y = o$, ergo axis ordinatarum ab utraque parte tangitur. Sit

$$\frac{b}{d} = \frac{m}{n} = 2, \quad a = c = 1,$$

inde evadit

$$x^4 + 9y^4 - 10x^2y^2 - 16d^2x^2 = o \tag{60.}$$

Itaque fit pro $y = o$

$$x = o, \quad x = \pm 4d$$

Asymptotis item aequationes (59.) satisfaciunt. Cf. fig. 11.

Denique curvarum rami inter se sejuncti sunt, si

$$\text{aut} \quad (\gamma.) \left\{ \begin{array}{l} \frac{b}{d} > \frac{m}{n} \\ \frac{a}{c} > \frac{m}{n} \end{array} \right. \qquad \text{aut} \quad (\delta.) \left\{ \begin{array}{l} \frac{b}{d} < \frac{m}{n} \\ \frac{a}{c} < \frac{m}{n} \end{array} \right.$$

Quum vero his conditionibus ratio inter $\frac{b}{d}$ et $\frac{a}{c}$ non pronunciata sit, investigemus, quid ab eadem pendeat. Jam si hoc animo contemplemur quantitatem $\frac{dy}{dx} =$

$$\frac{(m-n)[2mn(m-n)^2a^2c^2x^2 - mn(c^2m^2-a^2n^2)(a^2d^2-b^2c^2) \pm (m-n)(a^2n^2+c^2m^2)\sqrt{a^2c^2(m-n)^2x^2-(c^2m^2-a^2n^2)(a^2d^2-b^2c^2)}]}{(c^2m^2-a^2n^2)^2y\sqrt{a^2c^2(m-n)^2x^2-(c^2m^2-a^2n^2)(a^2d^2-b^2c^2)}} \tag{61}$$

quae infinitam capit valorem pro $y = o$ et pro

$$\left\{ \begin{array}{l} x' = \pm\sqrt{\frac{(c^2m^2 - a^2n^2)(a^2d^2 - b^2c^2)}{a^2c^2(m-n)^2}} \\ y' = \pm\sqrt{\frac{a^4n^2d^2 - c^4m^2b^2}{(c^2m^2 - a^2n^2)}}, \end{array} \right.$$

ratio illa inter $\frac{b}{d}$ et $\frac{a}{c}$ ita constituitur, ut curvarum tangentes ad abscissarum axem aut octo in punctis perpendiculares sint, si quidem conditio $(\gamma.)$ cum hac nova

$$(\varepsilon.) \quad \left\{ \begin{array}{l} \frac{b}{d} > \frac{a}{c} \\ \frac{a^2}{c^2} < \frac{bm}{dn} \end{array} \right.$$

vel conditio $(\delta.)$ cum hac altera

$$(\zeta.) \quad \left\{ \begin{array}{l} \frac{b}{d} < \frac{a}{c} \\ \frac{a^2}{c^2} > \frac{bm}{dn} \end{array} \right.$$

conjuncta est, aut in quatuor punctis, si res se aliter habet. E. g. ut conditionibus (γ.) et (ε.) simul satisfiat, si ponitur

$$\frac{b}{d} = 3, \quad \frac{m}{n} = \frac{1}{2}, \quad \frac{a}{c} = 1,$$

aeq. (53.) abit in

$$(62.) \qquad x^4 - 10x^2y^2 + 9y^4 - 74d^2x^2 + 210d^2y^2 + 1225d^4 = o$$

i. e.

$$9y^2 = 5x^2 - 105d^2 \pm 4x\sqrt{x^2 - 4d^2}$$

Curvae rami non existunt inter

$$\begin{cases} x = -2d\sqrt{6} \\ y = \pm d\sqrt{\frac{5}{3}} \end{cases} \text{ et } \begin{cases} x = +2d\sqrt{6} \\ y = \pm d\sqrt{\frac{5}{3}}, \end{cases}$$

et axi abscissarum occurrunt in punctis

$$x = \pm 5d \text{ et } x = \pm 7d,$$

Verum cum et in quatuor illis et in his sit $\frac{dy}{dx} = \infty$, curvae formam recte adumbratam esse in fig. 12, quisque intelligit. Asymptotarum aequationes sunt ut antea

$$\begin{cases} x \mp 3y = o \\ x \mp y = o \end{cases}$$

Sed ut aliud exemplum enumeremus, quod conditione (ε.) negata conditioni (γ.) soli conveniat, ponamus $\frac{b}{d} = \frac{1}{2}, \frac{m}{n} = \frac{1}{3}, \frac{a}{c} = 1$

Quo facto producitur aequatio

$$(63.) \qquad 16x^4 - 80x^2y^2 + 64y^4 - 104b^2x^2 + 80b^2y^2 + 25b^4 = o$$

sive

$$8y^2 = 5x^2 - 5b^2 \pm 3x\sqrt{x^2 + 6b^2}$$

ex qua cognoscitur, curvam axi abscissarum in punctis

$$x = \pm \frac{1}{2}b, \quad x = \pm \frac{5}{2}b$$

occurrere, atque in iisdem solis esse $\frac{dy}{dx} = \infty$. Asymptotae denotantur aequationibus

$$\begin{cases} x \mp 2y = o \\ x \mp y = o \end{cases}$$

Cf. fig. 13. in qua dimidia curvae pars delineata est.

Curvae illustratae in hyperbolas conicas transeunt si $\frac{b}{d} = \frac{a}{c}$, nam hac lege aeq. (53.) commutatur in

$$(64.) \quad o = \begin{cases} (m-n)^4x^4 + (c^2m^2 - a^2n^2)^2y^4 - 2(m-n)^2(a^2n^2 + c^2m^2)x^2y^2 \\ -2(a^2n^2 + c^2m^2)(n-m)^2\frac{d^2}{c^2}x^2 + 2(c^2m^2 - a^2n^2)\frac{d^2}{c^2}y^2 \\ + (c^2m^2 - a^2n^2)^2\frac{d^4}{c^4} \end{cases}$$

quae secerni potest hoc modo

$$\left\{\begin{array}{l}(cm - an)^2 y^2 - (m - n)^2 x^2 + (cm - an)^2 \dfrac{d^2}{c^2} = o \\ (cm + an)^2 y^2 - (m - n)^2 x^2 + (cm + an)^2 \dfrac{d^2}{c^2} = o\end{array}\right. \qquad (65.)$$

Exemplum singulare supersedemus commemorare; hinc igitur ad conditionem II $q = o$ transeuntes occurrimus aequationi

$$o = \left\{\begin{array}{l} x^4(c^2m^2 - a^2n^2)^2 - 2x^2y^2(m+n)^2(a^2n^2 + c^2m^2) + (m+n)^4 y^4 \\ - 2x^2(c^2m^2 - a^2n^2)(n^2b^2 - m^2d^2) - 2y^2(n^2b^2 + m^2d^2)(m+n)^2 \\ + (n^2b^2 - m^2d^2)^2 \end{array}\right. \qquad (66.)$$

quam, quum ad easdem fere curvas atque aeq. (53.) ducat, adumbrari opus non est. Quamobrem ad conditionem III

$$n^2(1 + a^2) - m^2(1 + c^2) = o$$

convertimur, unde provenit

$$o = \left\{\begin{array}{l} [x^4(m-n)^2 + y^4(m+n)^2 - 2x^2y^2(n^2 - m^2 + 2a^2n^2)][q^4(m-n)^2 - 2p^2q^2(n^2 - m^2 - 2a^2n^2) + p^4(m+n)^2] \\ -2x^2\left\{(n^2b^2 + m^2d^2)[q^2(m-n)^2 - p^2(n^2 - m^2 + 2a^2n^2)] + 2m^2n^2p^2[a^2d^2 + \dfrac{b^2}{m^2}(n^2 - m^2 + a^2n^2)]\right\} \\ -2y^2\left\{(n^2b^2 + m^2d^2)[p^2(m+n)^2 - q^2(n^2 - m^2 + 2a^2n^2)] + 2m^2n^2q^2[a^2d^2 + \dfrac{b^2}{m^2}(n^2 - m^2 + a^2n^2)]\right\} \\ + 8xy . pqm^2n^2\left[a^2d^2 - \dfrac{b^2}{m^2}(n^2 - m^2 + a^2n^2)\right] \\ + (n^2b^2 - m^2d^2)^2 \end{array}\right. \qquad (67)$$

Omnes curvas huic aequationi convenientes ramos infinitos octo habere judicandum est, nam membro summo $= o$ posito reperitur

$$y = \pm \frac{x}{m+n}\left[an \pm \sqrt{n^2 - m^2 + a^2n^2}\right],$$

cujus quantitatis valor, cum ex conditione III sit $n^2 - m^2 + a^2n^2 = m^2c^2$, semper realis est. Si ponimus e. g.

$$\frac{m}{n} = 1;\ p^2 = q^2 = \frac{1}{2};\ \ a^2 = c^2 = \frac{1}{2};\ \ b = 2,\ \ d = 1$$

inde evadit aequatio

$$2y^4 - x^2y^2 - 20y^2 - 6xy + 9 = o \qquad (68.)$$

sive

$$xy = -3 \pm \sqrt{2(y^2 - 9)(y^2 - 1)} \qquad (69.)$$

Membro summo aequationis (68.) $= o$ posito, producitur

$$\left\{\begin{array}{l} y = o \\ y \mp x\sqrt{\dfrac{1}{2}} = o; \end{array}\right. \qquad (70.)$$

ergo curva octo ramos infinitos habet. Quatuor eorum inter $y = -1$ et $y = +1$ sitos esse, duos vero ultra $y = +3$ et duos alteros ultra $y = -3$ porrigi ex aeq. (69.) apparet. Inter $y = \pm 1$ et $y = \pm 3$ curvae rami non existunt. In fig. 14 curvae rami in positiva ordinatarum parte siti delineati sunt.

Postremo adhuc memorandum est, aequationem (67.), si membrum xy evanescat, in duas hyperbolas conicas cadere. Quum enim hac lege ponendum sit

$$a^2d^2 - \frac{b^2}{m^2}(n^2 - m^2 + a^2n^2) = o,$$

illa aequatio commutatur in hanc

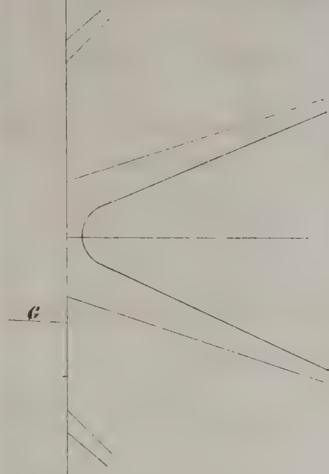
$$(71)\; o = \left\{\begin{array}{l} [x^4(m-n)^2 - 2x^2y^2(n^2-m^2+2a^2n^2) + y^4(m+n)^2][q^4(m-n)^2 - 2p^2q^2(n^2-m^2+2a^2n^2) + p^4(m+n)^2] \\ -2x^2\frac{b^2}{a^2}(n-m)^2[q^2(n^2-m^2+2a^2n^2) - p^2(n+m)^2] \\ -2y^2\frac{b^2}{a^2}(m+n)^2[p^2(n^2-m^2+2a^2n^2) - q^2(n-m)^2] \\ + \frac{b^4}{a^4}(n^2-m^2)^2 \end{array}\right.$$

quae in has duas solvi potest

$$(72.) \qquad o = \left\{\begin{array}{l} [y^2(m+n)^2 - x^2[an + \sqrt{n^2-m^2+a^2n^2}]^2][(m-n-2np^2)^2 - 4p^2q^2a^2n^2] \\ -\frac{b^2}{a^2}(m+n)^2[p^2(an+\sqrt{n^2-m^2+a^2n^2})^2 - q^2(n-m)^2] \end{array}\right.$$

$$(73.) \qquad o = \left\{\begin{array}{l} [y^2(m+n)^2 - x^2(an - \sqrt{n^2-m^2+a^2n^2})^2][(m-n-2np^2)^2 - 4p^2q^2a^2n^2] \\ -\frac{b^2}{a^2}(m+n)^2[p^2(an-\sqrt{n^2-m^2+a^2n^2})^2 - q^2(n-m)^2] \end{array}\right.$$

Denique aequatio (52.) integra adhuc disquirenda est. Omnes curvas in eam cadentes, ab utroque parte coordinatarum axe in dimidias partes secari, statim elucet, sed nihil amplius de iis in universum dicendum est. Nam ne id quidem dijudicari potest, num omnes habeant ramos infinitos; verum, si exstant, ad lineas rectas sese applicant, id quod demonstrari opus non est.



Fi

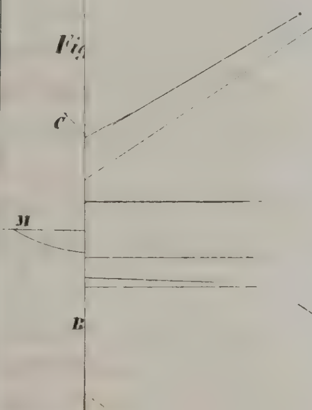
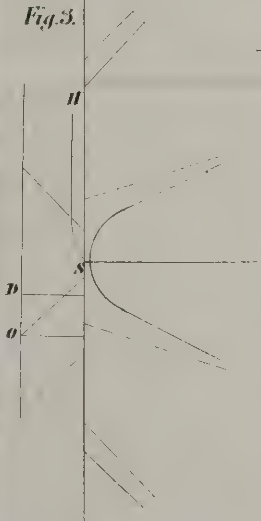


Fig.3.



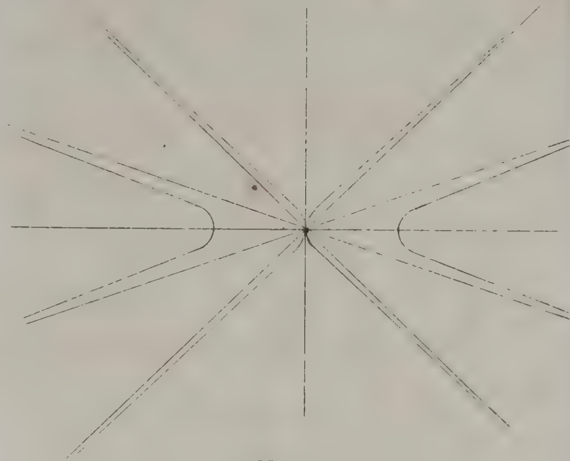


Fig. 11.

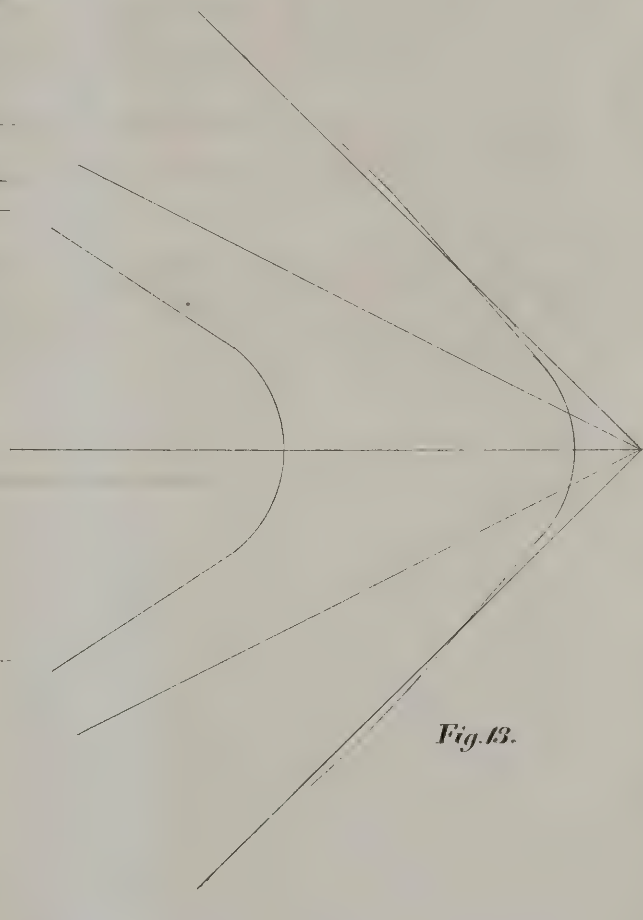
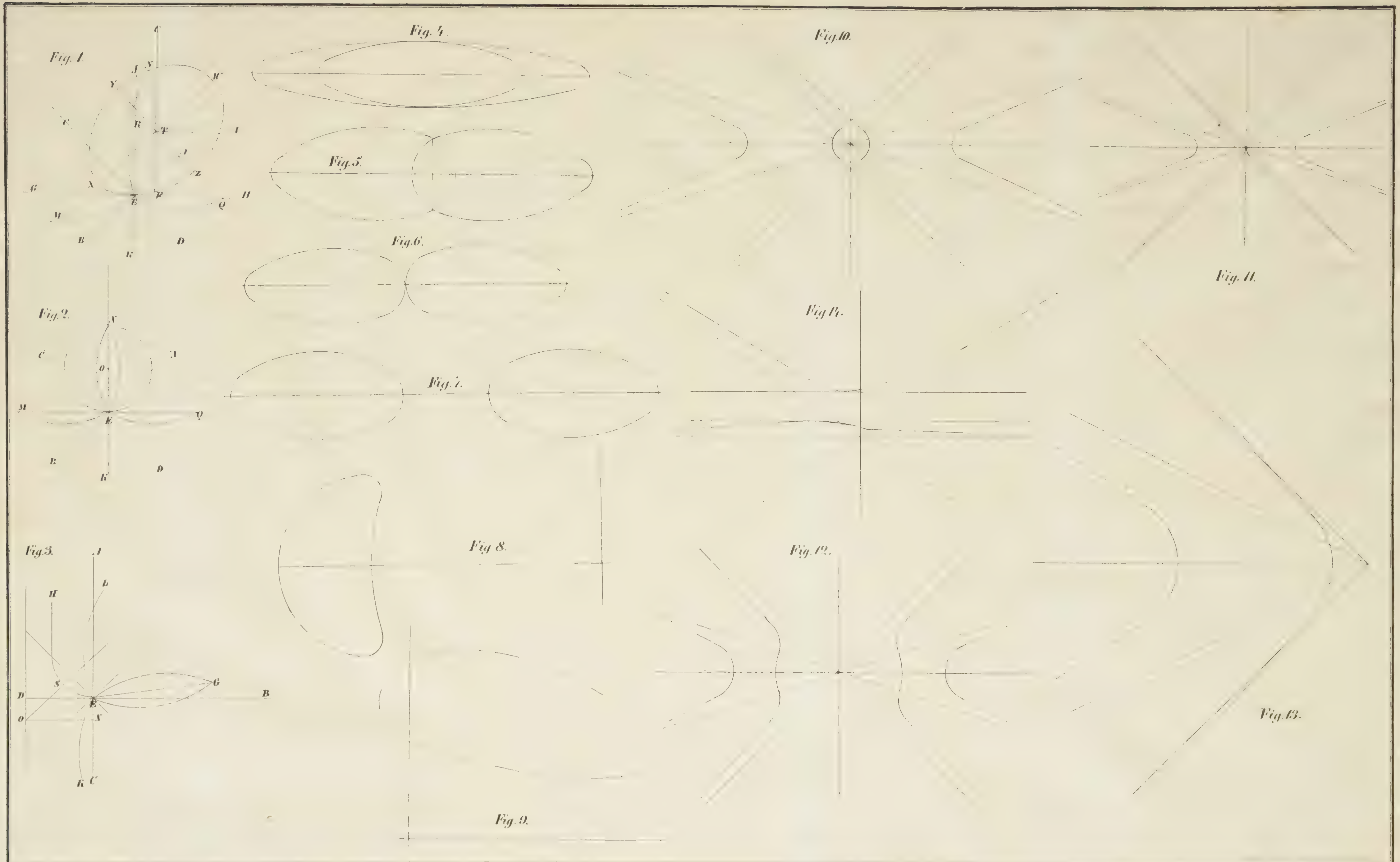


Fig. 13.


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.

VITA.

Natus sum Reichenbachii oppido in Silesia inferiore sito Calendis Februariis anno hujus saeculi nono, patre mercatore, viginti tribus abhinc annis defuncto, matre Rosina e gente Guttgeselliana, quam adhuc superstitem esse laetor. Domicilio inde, quum adhuc puerulus essem, Strehlas translato, elementis litterarum a viro summe venerando Maidornio, qui in hoc ipso oppido etiamnunc sacra curat, usque ad decimum quartum aetatis annum institutus sum. Tum vero vitricus meus, FERDINANDUS SCHOLZ, quem morte ereptum lugeo, Gymnasio hujus urbis Magdalenaeo, cui tum MANSO V. D. praeerat me commisit, ubi praeceptores doctissimi et dilectissimi de me maxime meruerunt. Post sexennium cum testimonio maturitatis dimissus ad hanc almam litterarum sedem me contuli, ibique per quatuor annos ad res mathematicas et physicas incubui. Mathesin me docuerunt JUNGNITZ, SCHOLTZ, KOECHER, Physicen STEFFENS et POHL, Astronomiam JUNGNITZ et SCHOLTZ, Viri Clarissimi Doctissimi. Neque tamen alia studia neglexi. In rebus theologicis me instituerunt SCHULZ et MIDDELDORPF, in philosophicis THILO et BRANISS, in philologicis PASSOW et SCHNEIDER, in historicis WACHLER et STENZEL, V. V. C. Quibus omnibus praeceptoribus dilectissimis gratias quam maximas ago, in pioque animo per omne tempus eos habebo. Verum in primis Viri Doctissimi Celeberrimi SCHOLTZ beneficia erga me plurima exstant, nam semper consilio optimo me rexit, seque in quacunque vitae conditione amicissimum mihi ita praebuit, ut ejus memoria animo meo nunquam excidat.

Academia Mense Martis anni MDCCCXXXIII relicta, Augusto ejusdem anni examen, quod vocant paedagogicum, subii. Tum in Seminarium Regium paedagogicum receptus quum in gymnasiis hujus urbis aliquamdiu pueros instituissem, anno praeterlapso, in quo ipse litteras edoctus sum gymnasio, Magdalenaeo, docendi munus accepi.

THESES.

I. Aër terrae circumfusus non mechanica sed chemica compositio est.

II. Stellae cadentes sunt corpora planetis omnino similia.

III. Solis atmosphaera neque ejus corpus solidum lucet.